

Berry Phase And Chern Number

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- 1 Berry Phase
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Berry Phase

Berry connection is

$$A(\lambda) = \langle u_\lambda | i \partial_\lambda u_\lambda \rangle = -\text{Im} \langle u_\lambda | \partial_\lambda u_\lambda \rangle \quad (1)$$

in terms of Berry phase is

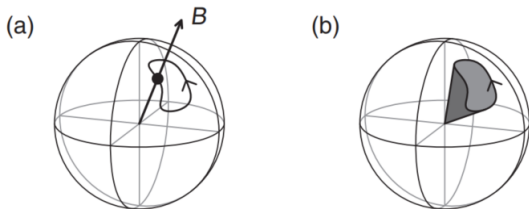
$$\phi = \oint A(\lambda) d\lambda \quad (2)$$

$$\phi = -\text{Im} \ln [\langle u_0 | u_1 \rangle \langle u_1 | u_2 \rangle \cdots \langle u_{N-1} | u_0 \rangle] \quad (3)$$

A spin- $\frac{1}{2}$ particle subjected to a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{n}}$ directed along $\hat{\mathbf{n}}$

$$H = -\gamma \mathbf{B} \cdot \mathbf{S} = -\left(\frac{\gamma \hbar B}{2}\right) \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \quad (4)$$

Berry Phase



$$\phi = -\text{Im} \ln [\langle \uparrow_z | \uparrow_x \rangle \langle \uparrow_x | \uparrow_y \rangle \langle \uparrow_y | \uparrow_z \rangle] \quad (5)$$

$$|\uparrow_{\hat{n}}\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{bmatrix} \quad (6)$$

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad |\uparrow_y\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \quad |\uparrow_z\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (7)$$

$$\phi = \frac{\pi}{4}$$

$$A_\mu = \langle u_\lambda | i \partial_\mu u_\lambda \rangle \quad (8)$$

$$\Omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -2 \text{Im} \langle \partial_\mu u | \partial_\nu u \rangle \quad (9)$$

$$\phi = \oint_P \mathbf{A} \cdot d\lambda = \int_S \Omega_{\mu\nu} ds_\mu \wedge ds_\nu \quad (10)$$

where $ds_\mu \wedge ds_\nu$ is an area element on the surface S and P is its boundary.

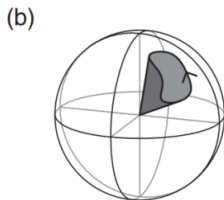
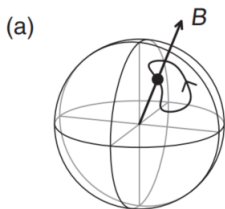
$$| \uparrow_{\hat{\mathbf{n}}} \rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\varphi} \end{bmatrix} \quad (11)$$

This representation is smooth and continuous in the vicinity of "north pole" ($\theta = 0$) of the Bloch sphere. Take $\lambda = (n_x, n_y)$ through

$$\hat{\mathbf{n}} = (n_x, n_y, \sqrt{1 - n_x^2 - n_y^2}).$$

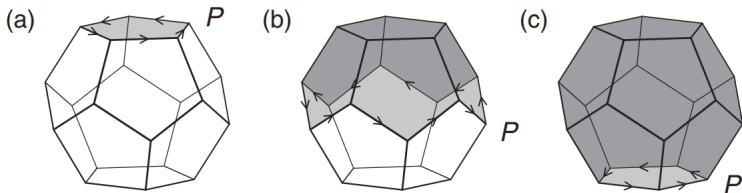
Berry Curvature

$$|\uparrow_{\hat{\mathbf{n}}}\rangle \approx \begin{bmatrix} 1 \\ (n_x + in_y)/2 \end{bmatrix} \quad |\partial_{n_x} \uparrow_{\hat{\mathbf{n}}}\rangle = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad |\partial_{n_y} \uparrow_{\hat{\mathbf{n}}}\rangle = \frac{1}{2} \begin{bmatrix} 0 \\ i \end{bmatrix} \quad (12)$$
$$\Omega_{xy} = -\frac{1}{2}$$



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Chern Theorem



$$(a) \rightarrow -\frac{\pi}{6} \quad (b) \rightarrow -\pi \quad (c) \rightarrow -\frac{11\pi}{6}$$

$$\oint_S \boldsymbol{\Omega} \cdot d\mathbf{S} = 2\pi C \quad (13)$$

When the Chern index is nonzero, it is impossible to construct a smooth and continuous gauge over the entire surface S .

Chern Theorem

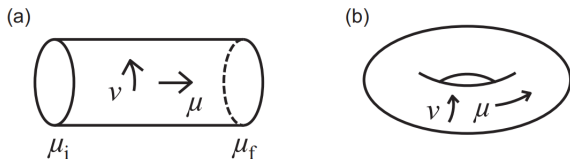
$$|\uparrow_{\hat{\mathbf{n}}}\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{bmatrix} \quad (14)$$

while it is smooth in the "northern hemisphere", this gauge has a singularity ("vortex") at $\theta = \pi$, the "south pole".

$$|\uparrow_{\hat{\mathbf{n}}}\rangle = \begin{bmatrix} \cos(\theta/2)e^{-i\varphi} \\ \sin(\theta/2) \end{bmatrix} \quad (15)$$

There is no possible choice of gauge that is smooth and continuous everywhere on the unit sphere. In such a case, we say that the presence of a nonzero Chern index presents a **topological obstruction** to the construction a globally smooth gauge.

Berry Flux on a Cylinder or Torus



Berry flux

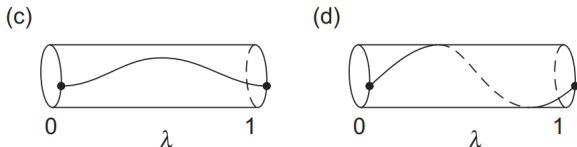
$$\Phi^{\mu\nu} = \oint_S \Omega_{\mu\nu} dS \quad (16)$$

For cylinder geometry

$$\begin{aligned} \Phi^{\mu\nu} &= \int_{\mu_i}^{\mu_f} d\mu \int_0^1 d\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \int_{\mu_i}^{\mu_f} (\partial_\mu \phi^\nu) d\mu = \phi^\nu(\mu_f) - \phi^\nu(\mu_i) \end{aligned} \quad (17)$$

Berry Flux on a Cylinder or Torus

In the case $\mu = 0$ and $\mu = 1$ are identified, the Berry phase at this two end points must match modulo 2π . so that at the end of the cycle on the μ , ϕ^ν must have evolved by $2\pi m$ for some integer m .



The Chern number is nothing other than the winding number of the Berry phase along ν as we evolve around a cycle in μ .

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Chern Insulator

$$H_{CI}(\mathbf{k}) = \lambda_x \sin k_x \sigma_x + \lambda_y \sin k_y \sigma_y + (m_0 + t_x \cos k_x - t_y \cos k_y) \sigma_z \quad (18)$$

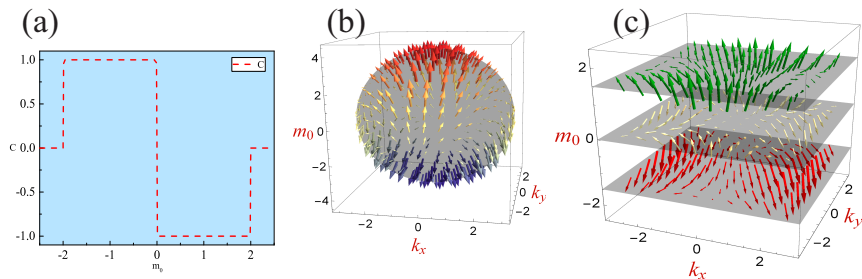


Figure: (a) Chern number of H_{CI} . (b-c) Vector plot for the components $(\sin k_x, \sin k_y, (m_0 + t_x \cos k_x - t_y \cos k_y))$. Common parameters: $\lambda_x = \lambda_y = 1.0, t_x = t_y = 1.0$

Chern Insulator

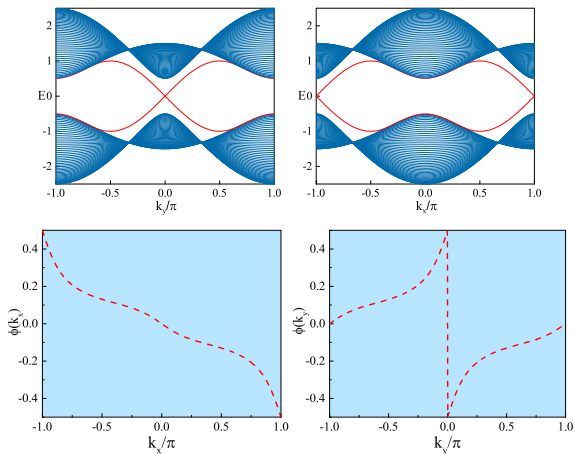


Figure: $m_0 = 0.5, \lambda_x = \lambda_y = 1.0, t_x = t_y = 1.0$