Berry Phase And Chern Number

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Outline

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Berry connection is

$$A(\lambda) = \langle u_\lambda | i\partial_\lambda u_\lambda \rangle = -\mathrm{Im}\langle u_\lambda | \partial_\lambda u_\lambda \rangle \tag{1}$$

in terms of Berry phase is

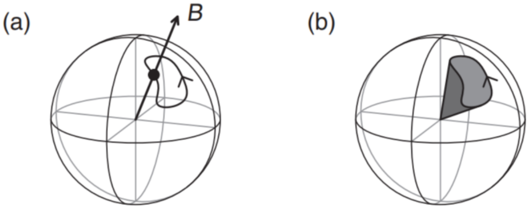
$$\phi = \oint A(\lambda) d\lambda \tag{2}$$

$$\phi = -\mathrm{Im}\ln\left[\langle u_0|u_1\rangle\langle u_1|u_2\rangle \cdots \langle u_{N-1}|u_0\rangle\right] \tag{3}$$

A spin-$\frac{1}{2}$ particle subjected to a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{n}}$ directed along $\hat{\mathbf{n}}$

$$H = -\gamma \mathbf{B}\cdot\mathbf{S} = -(\frac{\gamma\hbar B}{2})\hat{\mathbf{n}}\cdot\sigma \tag{4}$$

Berry Phase



$$\phi = -\mathrm{Im}\ln\left[\langle\uparrow_z \mid \uparrow_x\rangle\langle\uparrow_x \mid \uparrow_y\rangle\langle\uparrow_y \mid \uparrow_z\rangle\right] \tag{5}$$

$$|\uparrow_{\hat{\mathbf{n}}}\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{bmatrix} \tag{6}$$

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad |\uparrow_y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} \quad |\uparrow_z\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{7}$$

$$\phi = \frac{\pi}{4}$$

Berry Curvature

$$A_\mu = \langle u_\lambda | i\partial_\mu u_\lambda \rangle \tag{8}$$

$$\Omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -2\mathrm{Im}\langle \partial_\mu u | \partial_\nu u \rangle \tag{9}$$

$$\phi = \oint_P \mathbf{A} \cdot d\lambda = \int_S \Omega_{\mu\nu} ds_\mu \wedge ds_\nu \tag{10}$$

where $ds_\mu \wedge ds_\nu$ is an area element on the surface S and P is its boundary.

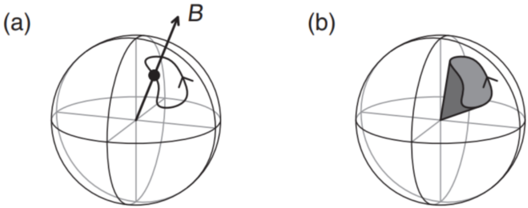
$$|\uparrow_{\hat{\mathbf{n}}}\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{bmatrix} \tag{11}$$

This representation is smooth and continuous in the vicinity of "north pole"$(\theta = 0)$ of the Bloch sphere. Take $\lambda = (n_x, n_y)$ through $\hat{\mathbf{n}} = (n_x, n_y, \sqrt{1 - n_x^2 - n_y^2})$.

Berry Curvature

$$|\uparrow_{\hat{\mathbf{n}}}\rangle \approx \begin{bmatrix} 1 \\ (n_x + i n_y)/2 \end{bmatrix} \quad |\partial_{n_x} \uparrow_{\hat{\mathbf{n}}}\rangle = \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad |\partial_{n_y} \uparrow_{\hat{\mathbf{n}}}\rangle = \frac{1}{2}\begin{bmatrix} 0 \\ i \end{bmatrix} \tag{12}$$

$$\Omega_{xy} = -\frac{1}{2}$$



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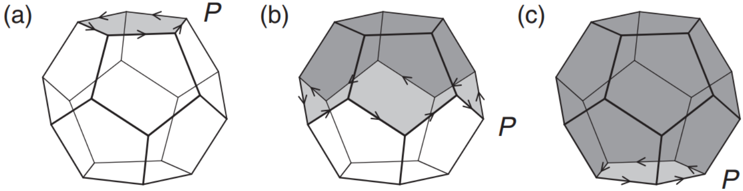
Chern Theorem



$$(a) \rightarrow -\frac{\pi}{6} \quad (b) \rightarrow -\pi \quad (c) \rightarrow -\frac{11\pi}{6}$$

$$\oint_S \boldsymbol{\Omega} \cdot d\mathbf{S} = 2\pi C \tag{13}$$

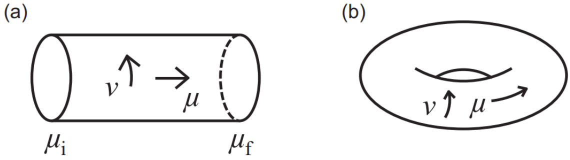
When the Chern index is nonzero, it is impossible to contruct a smooth and continuous gauage over the entire surface S.

$$|\uparrow_{\hat{\mathbf{n}}}\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\varphi} \end{bmatrix} \quad (14)$$

while it is smooth in the "northern hemisphere", this guage has a singularity("vortex") at $\theta = \pi$, the "south pole".

$$|\uparrow_{\hat{\mathbf{n}}}\rangle = \begin{bmatrix} \cos(\theta/2)e^{-i\varphi} \\ \sin(\theta/2) \end{bmatrix} \quad (15)$$

There is no possible choice of guage that is smooth and continuous everywhere on the unit sphere. In such a case, we say that the presence of a nonzero Chern index presents a topological obstruction to the construction a globally smooth gauge.



Berry flux

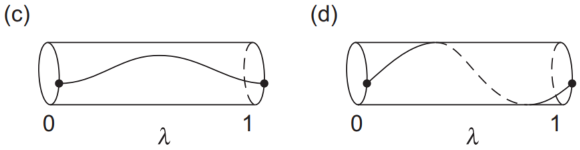
$$\Phi^{\mu\nu} = \oint_S \Omega_{\mu\nu} dS \tag{16}$$

For cylinder geometry

$$\begin{aligned}\Phi^{\mu\nu} &= \int_{\mu_i}^{\mu_f} d\mu \int_0^1 d\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= \int_{\mu_i}^{\mu_f} (\partial_\mu \phi^\nu) d\mu = \phi^\nu(\mu_f) - \phi^\nu(\mu_i)\end{aligned} \tag{17}$$

Berry Flux on a Cylinder or Torus

In the case $\mu = 0$ and $\mu = 1$ are indetified, the Berry phase at this two end points must match modulo 2π. so that at the end of the crycle on the μ, ϕ^{ν} must have evolved by $2\pi m$ for some integer m.



The Chern number is nothing other than the winding number of the Berry phase along ν as we evolve around a cycle in μ.

Outline

$$H_{CI}(\mathbf{k}) = \lambda_x \sin k_x \sigma_x + \lambda_y \sin k_y \sigma_y + (m_0 + t_x \cos k_x - t_y \cos k_y)\sigma_z \quad (18)$$

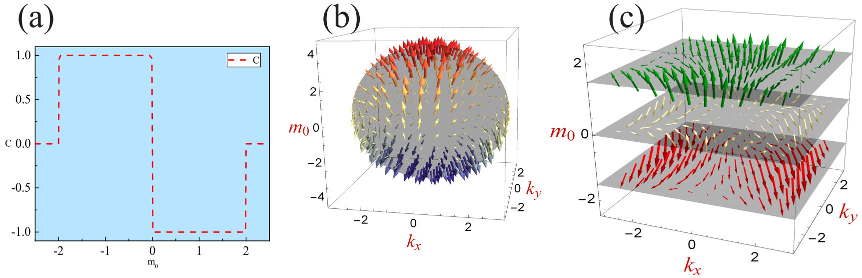


Figure: (a) Chern number of H_{CI} . (b-c) Vector plot for the components $(\sin k_x, \sin k_y, (m_0 + t_x \cos k_x - t_y \cos k_y))$. Common parameters: $\lambda_x = \lambda_y = 1.0, t_x = t_y = 1.0$

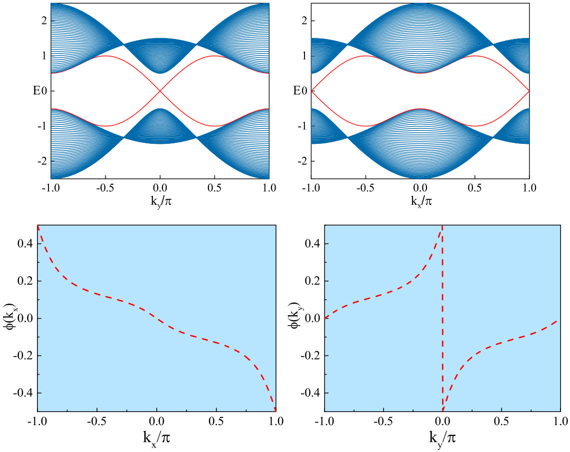


Figure: $m_0 = 0.5, \lambda_x = \lambda_y = 1.0, t_x = t_y = 1.0$