Non-Hermitian topological

(To throw out a brick to attract a jade)

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Outline

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Eigenvalue and Eigenvector

$$M\mathbf{v} = \lambda\mathbf{v}, \lambda \in \mathcal{C} \tag{1}$$

For an arbitary polynomial $p(x) = \sum_{l=1}^{N} c_l x^l$, we have

$$p(M)\mathbf{v} \equiv \sum_{l=1}^{N} c_l M^l \mathbf{v} = \sum_{l=1}^{N} c_l \lambda^l \mathbf{v} = p(\lambda)\mathbf{v} \tag{2}$$

To determine the eigenvalues of matrix

$$(\lambda I - M)\mathbf{v} = 0 \tag{3}$$

All the possible eigenvalues λ's can ben determined from

$$p_M(\lambda) \equiv \det(\lambda I - M) = 0 \tag{4}$$

where $p_M(\lambda)$ is called the characteristic polynomial of M.

Eigenvalue and Eigenvector

Generally decompose $p_M(\lambda)$ into

$$p_M(\lambda) = \prod_{j=1}^{J} (\lambda - \lambda_j)^{m_j^a} \tag{5}$$

where $\lambda_j \neq \lambda_{j'}$ for $\forall j \neq j'$, $\sum_{j=1}^{J} m_j^a = n$. The sepctrum of M, denoted as $\Lambda(M)$ is defined as

$$\Lambda(M) \equiv \bigcup_{j=1}^{j} \{\lambda_j\}^{\cup m_j^a} \tag{6}$$

Each element in the $\Lambda(M)$ is real for a Hermitian matrix M, since

$$\lambda = \frac{\mathbf{v}^\dagger M \mathbf{v}}{\mathbf{v}^\dagger \mathbf{v}} = \frac{\mathbf{v}^\dagger M^\dagger \mathbf{v}}{\mathbf{v}^\dagger \mathbf{v}} = \lambda^* \tag{7}$$

Eigenvalue and Eigenvector

We define the eigenspace associated with λ_j as

$$\mathbf{V}_m(\lambda_j) \equiv \mathrm{Ker}(M - \lambda_j I) \equiv \mathrm{span}\{\mathbf{v}_j : M\mathbf{v}_j = \lambda_j \mathbf{v}_j, \mathbf{v}_j \in \mathcal{C}^n\}$$

Then its dimension $m_j^g \equiv \dim \mathbf{V}_M(\lambda_j)$ must be a positive integer. For any eigenvalue λ_j, we have

$$m_j^g \leq m_j^a \tag{8}$$

$\mathbf{V}_M(\lambda)$ is an invariant subspace of M(For any vector $\mathbf{v}$ in the subspace, $M\mathbf{v}$ still lies in the same subspace.), $p_M(\lambda)$ must contain a factor $(\lambda - \lambda_j)^{m_j^g}$

If M is Hermitian, we have $m_i^g = m_i^a$.
If M is non-Hermitian, things are much more complicated in general.

Example for non-Hermitian matrix

We consider

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \tag{9}$$

which hase the eigenvector $\mathbf{v} = (1,0)^T$ with a double degenerate eigenvalue λ. In this case, we have $m^a = 2 > m^g = 1$.

Eigenvectors of non-Hermitian matrices

Given $\mathbf{v}_1$ and $\mathbf{v}_2$ as two eigenvectors of a Hermitian matrix M with different eigenvalues $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1^\dagger \mathbf{v}_2 = 0$, this is because

$$\mathbf{v}_1^\dagger M \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^\dagger \mathbf{v}_2 = (M\mathbf{v}_1)^\dagger \mathbf{v}_2 = \lambda_1 \mathbf{v}_1^\dagger \mathbf{v}_2 \rightarrow (\lambda_1 - \lambda_2)\mathbf{v}_1^\dagger \mathbf{v}_2 = 0 \tag{10}$$

To generalize the orthogonality to non-Hermitian matrices, we should require $\mathbf{v}_1$ to be an eigenvector of $M^\dagger$ with eigenvalue λ_1^*

$$M^\dagger \mathbf{v}_1 = \lambda_1^* \mathbf{v}_1 \Leftrightarrow \mathbf{v}_1^\dagger M = \lambda_1 \mathbf{v}_1^\dagger \tag{11}$$

so that we agine have

$$\mathbf{v}_1^\dagger M \mathbf{v}_2 = (M^\dagger \mathbf{v}_1)^\dagger \mathbf{v}_2 = \lambda_1 \mathbf{v}_1^\dagger \mathbf{v}_2 \tag{12}$$

An eigenvector of $M^\dagger$ like $\mathbf{v}_1$ in Eq.(11) as a left eigenvector of M, while the conventional one a right eigenvector.

Eigenvectors of non-Hermitian matrices

$$H|u_{R\alpha}\rangle = E_\alpha|u_{R\alpha}\rangle, \quad H^\dagger|u_{L\alpha}\rangle = E^*|u_{L\alpha}\rangle \tag{13}$$

right eigenvector:

$$|u_{R\alpha}\rangle$$

left eigenvector:

$$|u_{L\alpha}\rangle$$

If diagonalize $H = VJV^{-1}$, J being diagonal, then every column of V(or $(V^\dagger)^{-1}$) is a right(or left) eigenvector, with the normalization $\langle u_{L\alpha}|u_{R\beta}\rangle = \delta_{\alpha\beta}$

Pseudo Hermitian

The spectrum is trivially real in a Hermitian matrix. However, the Hermiticity is not a necessary condition for eigenvalues to be real.

$$\text{Hermitian} \Rightarrow \text{Real eigenvalues}$$

$$\text{Real eigenvalues} \Rightarrow \text{Hermitian} \qquad (\times)$$

An operator M is said to be the η-pseudo-Hermitian if it satifies

$$M^\dagger = \eta M \eta^{-1} \tag{14}$$

where $\eta = \eta^\dagger$ is a Hermitian invertible operator.

Choose $\eta = I$, η-pseudo-Hermiticity reduces to an ordinary Hermiticity $M = M^\dagger$

Theorem

A linear operator M acting on the Hilbert space with a complete biorthonormal eigenbasis and a discrete spectrum is pseudo-Hermitian if and only if one of the following conditions hold:

- *The spectrum of M is entirely real*
- *The eigenvalues appear in complex conjugate pairs and the degeneracy of complex conjugate eigenvalues are the same*

Example for PT-symmetry

Consider a single quantum particle subject to a one-dimensional PT-symmetric complex potential. It is described by a non-Hermitian Hamiltonian

$$H = p^2 + V_r(x) + iV_i(x) \qquad V_r(x) = V(-x), V_i(x) = -V_i(-x) \tag{15}$$

where

$$PxP^{-1} = -x, PpP^{-1} = -p, TxT^{-1} = x, TpT^{-1} = -p, TiT^{-1} = -i$$
$$(PT)H(PT)^{-1} = H$$

Since the time-reversal operator T acts as complex conjugation, we arrive at

$$PHP^{-1} = H^\dagger \tag{16}$$

The Hamiltonian H is P-pseudo-Hermitian,its eigenvalues must either be real or from complex conjugate pairs.

PT transition

$$(PT)H(PT)^{-1} = H \quad H|u_\alpha\rangle = E|u_\alpha\rangle \tag{17}$$

PT unbroken:

$$(PT)|u_\alpha\rangle = \lambda|u_\alpha\rangle \tag{18}$$

$$|u_\alpha\rangle \in \text{PT-symmetry} \Rightarrow E \in \mathcal{R}$$

PT spontaneously broken:

$$(PT)|u_\alpha\rangle \neq \lambda|u_\alpha\rangle \tag{19}$$

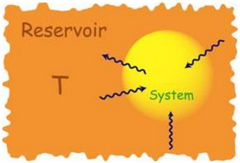
$$|u_\alpha\rangle \in \text{PT-symmetry} \Rightarrow (E, E^*) \notin \mathcal{C}$$

This real-to-complex spectral transition is often called the PT transition[1].

[1] Nat. Phys.14,11-19(2018)

Open Quantum Systems

The theory of open quantum systems describes the interaction of a quantum system with its environment



Quantum Mechanics

closed systems

unitary dynamics

reversible dynamics

$$i\hbar \frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle$$

Schrödinger Equation

$$i\hbar \frac{d\hat{\rho}}{dt} = [\hat{H}, \hat{\rho}]$$

Liouville – von Neumann Equation

open quantum systems

reduced density operator

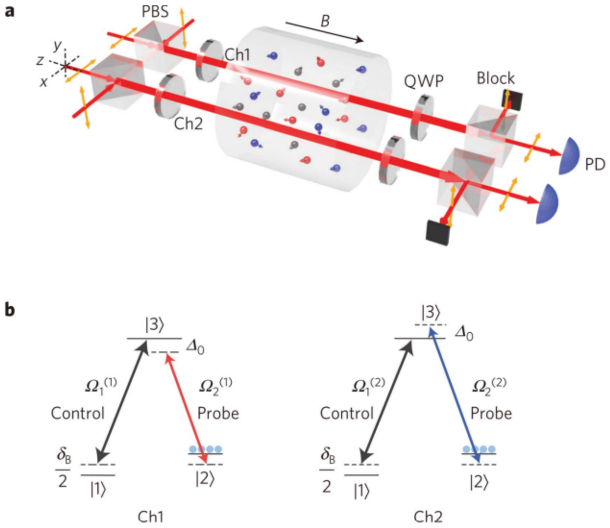
$$\hat{\rho}_S(t) = Tr_E[\hat{\rho}_T(t)]$$

master equation

$$\frac{d\hat{\rho}}{dt} = L\hat{\rho}$$

non-unitary and irreversible dynamics

Non-Hermitian System[2]



[2]: Zhaoyang Zhang et al 2018 J. Phys. B: At. Mol. Opt. Phys. 51 072001

Outline

SSH Model

Single-particle Hamiltonian is expressed as

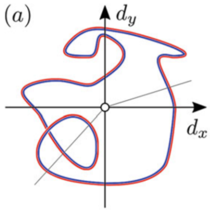
$$H = v\sum_{m=1}^{N}(|m,B\rangle\langle m,A| + h.c) + w\sum_{m=1}^{N-1}(|m+1,A\rangle\langle m,B| + h.c) \tag{20}$$

$$H(k) = \mathbf{d}(k)\hat{\sigma} \tag{21}$$

where $d_x = v + w\cos(k)$, $d_y = w\sin(k)$, $d_z = 0$

$$E(k) = \pm\sqrt{v^2 + w^2 + 2vw\cos(k)} \tag{22}$$

Winding number



The winding number v is given by

$$v = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\tilde{\mathbf{d}}(k) \times \frac{d}{dk} \tilde{\mathbf{d}}(k))_z dk \tag{23}$$

where

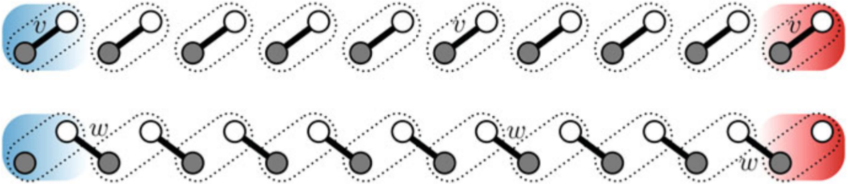
$$\tilde{\mathbf{d}} = \frac{\mathbf{d}}{|\mathbf{d}|}$$

Winding number

$$H(k) = \begin{pmatrix} 0 & h(k) \\ h(k)^* & 0 \end{pmatrix}; \qquad h(k) = d_x(k) - id_y(k) \tag{24}$$

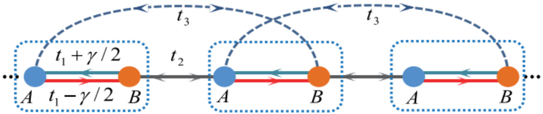
Using the complex logarithm function $\log(|h|e^{i\arg h}) = \log|h| + i\arg h$, we have

$$v = \frac{1}{2\pi i}\int_{-\pi}^{\pi} dk \frac{d}{dk}\log h(k) \tag{25}$$



Outline

Non-Hermitian SSH Model[3]



$$
\begin{aligned}
H(k) &= d_x\sigma_x + (d_y + i\frac{\gamma}{2})\sigma_y \\
E(k) &= \sqrt{d_x^2 + (d_y + i\frac{\gamma}{2})^2}
\end{aligned} \tag{26}
$$

$$d_x = t_1 + (t_2+t_3)\cos(k) \qquad d_y = (t_2 - t_3)\sin(k)$$

Phase transition: $E(k) = 0$

Take $t_3 = 0$, we can derive $(d_x, d_y) = (\pm\gamma/2, 0)$ which require $t_1 = t_2 \pm \frac{\gamma}{2}(k=\pi)$ or $t_1 = -t_2 \pm \frac{\gamma}{2}(k=0)(\times)$

[3] S.Yao,Z.Wang,PRL,121,086803

Gap close

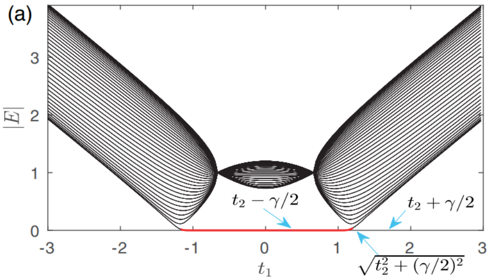


Figure: Open boundary energy band

Note

- $E(k) = 0$ do not correspond phase transition, bulk-boundary correspond is failure.

```julia
function matrixSet(xn::Int64,t1::Float64,t2::Float64,t3::Float64,gam::Float64)
    ham = zeros(ComplexF64,xn*2,xn*2)
    sigx = zeros(Float64,2,2)
    sigx[1,2] = 1.0
    sigx[2,1] = 1.0
    sigy = zeros(ComplexF64,2,2)
    sigy[1,2] = -1im
    sigy[2,1] = 1im
    #---------------------------------
    for k in 0:xn-1
        if k == 0    # First line
           for m1 in 1:2
                for m2 in 1:2
                    ham[m1,m2] = t1*sigx[m1,m2] + 1im*gam/2.0*sigy[m1,m2]
                    ham[m1,m2 + 2] = (t2 + t3)/2.0*sigx[m1,m2] - 1im*(t2 - t3)/2.0*sigy[m1,m2]
                end
            end
        elseif k == xn-1
            for m1 in 1:2
                for m2 in 1:2
                    ham[k*2 + m1,k*2 + m2] = t1*sigx[m1,m2] + 1im*gam/2.0*sigy[m1,m2]
                    ham[k*2 + m1,k*2 + m2 - 2] = (t2 + t3)/2.0*sigx[m1,m2] + 1im*(t2 - t3)/2.0*sigy[m1,m2]
                end
            end
        else
            for m1 in 1:2
                for m2 in 1:2
                    ham[k*2 + m1,k*2 + m2] = t1*sigx[m1,m2] + 1im*gam/2.0*sigy[m1,m2]
                    #   right hopping
                    ham[k*2 + m1,k*2 + m2 + 2] = (t2 + t3)/2.0*sigx[m1,m2] - 1im*(t2 - t3)/2.0*sigy[m1,m2]
                    #   left hopping
                    ham[k*2 + m1,k*2 + m2 - 2] = (t2 + t3)/2.0*sigx[m1,m2] + 1im*(t2 - t3)/2.0*sigy[m1,m2]
                end
            end
        end
    end
    #---------------------------------
    return ham
end
```

Similarity transformation

$$|\psi\rangle = (\psi_{1,A}, \psi_{1,B}, \psi_{2,A}, \psi_{2,B}, \cdots, \psi_{L,A}, \psi_{L,B})^T$$

$$H|\psi\rangle = E|\psi\rangle \Leftrightarrow \bar{H}|\bar{\psi}\rangle = E|\bar{\psi}\rangle \quad \text{with} \quad |\bar{\psi}\rangle = S^{-1}|\psi\rangle \tag{27}$$

$$\bar{H} = S^{-1}HS \tag{28}$$

$$S = \{1, r, r, r^2, r^2, \cdots, r^{L-1}, r^{L-1}, r^L\}$$

For $\bar{H}$: $t_1 \pm \frac{\gamma}{2} \to r^{\pm 1}(t_1 + \pm\frac{\gamma}{2})$, if take $r = \sqrt{|(t_1 - \gamma/2)/(t_1 + \gamma/2)|}$, $\bar{H}$ becomes the standard SSH model for $|t_1| > |\gamma/2|$, with intracell and intercell hoppings

$$\bar{t}_1 = \sqrt{(t_1 - \gamma/2)(t_1 + \gamma/2)}, \quad \bar{t}_2 = t_2 \tag{29}$$

Gap close

$$\bar{H}(k) = (\bar{t}_1 + \bar{t}_2 \cos(k))\sigma_x + \bar{t}_2 \sin(k)\sigma_y \tag{30}$$

Phase transition: $E(k) = 0 \rightarrow \bar{t}_1 = \bar{t}_2$

$$t_1 = \pm\sqrt{t_2^2 + (\gamma/2)^2} \qquad (\checkmark) \tag{31}$$

Note

With the help of similarity transformation, Non-Hermitian SSH model can be modified to Hermitian SSH model, energy gap closure could perdict phase transition.

Generalizable solution

Real space eigen-equation is

$$\begin{aligned} t_2\psi_{n-1,B} + [t_1 + (\gamma/2)]\,\psi_{n,B} =& sE\psi_{n,A} \\ [t_1 - (\gamma/2)]\,\psi_{n,A} + t_2\psi_{n+1,A} =& E\psi_{n,B} \end{aligned} \tag{32}$$

Take the ansatz that $|\psi\rangle = \sum_j |\phi^{(j)}\rangle$, where each $|\phi^{(j)}\rangle$ takes the exponential form: $(\phi_{n,A}, \phi_{n,B}) = \beta^n(\phi_A, \phi_B)$, which satisfies

$$\begin{aligned} \left[t_1 + \frac{\gamma}{2} + t_2\beta^{-1}\right]\phi_B =& E\phi_A \\ \left[(t_1 - \frac{\gamma}{2}) + t_2\beta\right]\phi_A =& E\phi_B \end{aligned} \tag{33}$$

Therefore, we have

$$\left[(t_1 - \frac{\gamma}{2}) + t_2\beta\right]\left[(t_1 + \frac{\gamma}{2}) + t_2\beta^{-1}\right] = E^2 \tag{34}$$

Generalizable solution

two solutions

$$\beta_{1,2}(E) = \left[E^2 + \gamma^2/4 - t_1^2 + t_2^2 \pm \sqrt{(E^2 + \gamma^2/4 - t_1^2 - t_2^2)^2 - 4t_2^2(t_1^2 - \gamma^2/4)}\right] / \left[2t_2(t_1 + \gamma/2)\right] \tag{35}$$

In the $E \to 0$ limit, we have

$$\beta_{1,2}^{E\to 0} = -\frac{t_1 - \gamma/2}{t_2}, \quad -\frac{t_2}{t_1 + \gamma/2} \tag{36}$$

These two solutions correspond to $\phi_B = 0$ and $\phi_A = 0$, respectively.
Restoring the j index $|\phi^{(j)}\rangle$, we have

$$\phi_A^{(j)} = \frac{E}{t_1 - \gamma/2 + t_2\beta}\phi_B^{(j)}, \quad \phi_B^{(j)} = \frac{E}{t_1 + \gamma/2 + t_2\beta^{-1}}\phi_A^{(j)} \tag{37}$$

Generalizable solution

The general solution is writted as a linear combination

$$\begin{pmatrix} \Psi_{n,A} \\ \Psi_{n,B} \end{pmatrix} = \beta_1^n \begin{pmatrix} \phi_A^{(1)} \\ \phi_B^{(1)} \end{pmatrix} + \beta_2^n \begin{pmatrix} \phi_A^{(2)} \\ \phi_B^{(2)} \end{pmatrix} \tag{38}$$

Boundary condition

$$\begin{aligned} (t_1 + \gamma/2)\Psi_{1,B} - E\Psi_{1,A} =& 0 \\ (t_1 - \gamma/2)\Psi_{L,A} - E\Psi_{L,B} =& 0 \end{aligned} \tag{39}$$

Together with (37), they lead

$$\beta_1^{L+1}(t_1 - \gamma/2 + t_2\beta_2) = \beta_2^{L+1}(t_1 - \gamma/2 + t_2\beta_1) \tag{40}$$

We are concerned about the spectrum for a long chain, which necessitates $|\beta_1| = |\beta_2|$ for the bulk eigenstates.

Generalizable solution

Combined with $\beta_1\beta_2 = (t_1 - \gamma/2)/(t_1 + \gamma/2)$, $|\beta_1| = |\beta_2|$ leads to

$$|\beta_j| = r \equiv \sqrt{\left|\frac{t_1 - \gamma/2}{t_1 + \gamma/2}\right|} \tag{41}$$

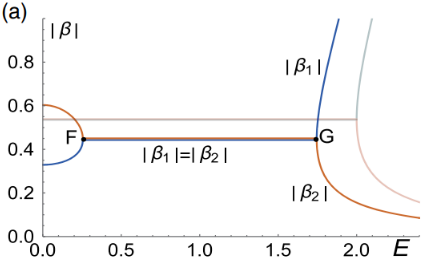


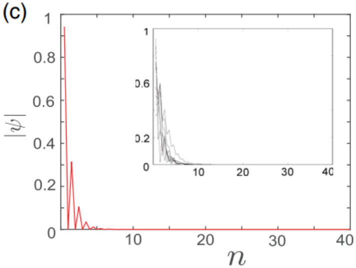
Figure: The excepted $|\beta_1| = |\beta_2| = r$ relation is found on the line FG, which is associated with bulk spectra.

Generalizable solution

Take $\beta = re^{ik}(k \in [0, 2\pi])$ into (34) to obtain the spectra

$$E^2(k) = t_1^2 + t_2^2 - \gamma^2/4 + t_2^2\sqrt{|t_1^2 - \gamma^2/4|}\left[\text{sgn}(t_1 + \gamma/2)e^{ik} + \text{sgn}(t_1 - \gamma/2)e^{-ik}\right] \tag{42}$$

whihc recovers the spectrum of SSH model when $\gamma = 0$. The phase transition points can be predicted when $|E(k)| = 0$



The usual Bloch wave carry a pure phase factor e^{ik}, whose role is now played by β. In addition to the phase factor, β has a modulus $|\beta| \neq 1$ in general.

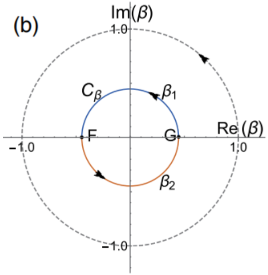
Outline

Winding Number

We start from the non-Bloch Hamiltonian obtained from $H(k)$ by the replacement $e^{ik} \to \beta$, $e^{-ik} \to \beta^{-1}$:

$$H(\beta) = (t_1 - \frac{\gamma}{2} + \beta t_2)\sigma_- + (t_1 + \frac{\gamma}{2} + \beta^{-1} t_2)\sigma_+ \tag{43}$$

where $\sigma + \pm = (\sigma_x \pm i\sigma_y)/2$, β takes values in a nonunit circle $|\beta| = r$(in other words,k acquires an imaginary part $-i \ln r$)



$$H(\beta)|u_R\rangle = E(\beta)|u_R\rangle, \quad H^\dagger(\beta)|u_L\rangle = E^*(\beta)|u_L\rangle \tag{44}$$

Chiral symmetry ensures that $|\tilde{u}_R\rangle \equiv \sigma_z|u_R\rangle$ and $|\tilde{u}_L\rangle \equiv \sigma_z|u_L\rangle$. $H(\beta) = TJT^{-1}$ with $J = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$, and the normalization condition $\langle u_L|u_R\rangle = \langle\tilde{u}_L|\tilde{u}_R\rangle = 1$, $\langle u_L|\tilde{u}_R\rangle = \langle\tilde{u}_L|u_R\rangle = 0$
Q matrix can be expressed as:

$$Q(\beta) = |\tilde{u}_R(\beta)\rangle\langle\tilde{u}_L(\beta)| - |u_R(\beta)\rangle\langle u_L(\beta)| \tag{45}$$

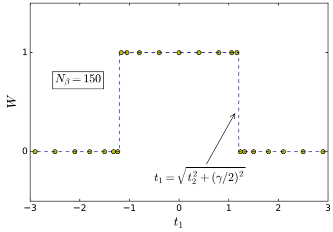
which is off-diagonal due to chiral symmetry $\sigma_z^{-1}Q\sigma_z = -Q$, namely, $Q = \begin{pmatrix} 0 & q \\ q^{-1} & 0 \end{pmatrix}$

```julia
# ========================================================
function Qmat(tt::Float64,tv::Float64)
    occ::Int64 = 1
    sigz = zeros(Float64,2,2)
    sigz[1,1] = 1
    sigz[2,2] = -1
    h1 = hamset(tt,tv)
    vecR = eigvecs(h1)
    vecL = inv(vecR)'
    q1 = vecR[:,occ]
    q11 = sigz*vecR[:,occ]
    q2 = vecL[:,occ]
    q22 = sigz*vecL[:,occ]
    Q = q11*q22' - q1*q2'
    return Q
end
# ========================================================
function winding(tv::Float64)
    re1::ComplexF64 = 0 + 0im
    kn::Int64 = 150
    qmlist = []
    qlist = []
    dq = []
    for k in 0:kn
        k = 2*pi/kn*k
        Q = Qmat(k,tv)
        append!(qlist,Q[1,2])
        append!(qmlist,Q[2,1])
    end
    #----------------------
    dq = qlist[2:end] - qlist[1:end-1]
    for k in 1:length(qmlist)-1
        re1 += qmlist[k]*dq[k]
    end
    return re1*1im/(2*pi)
end
# ========================================================
```

Winding Number

Non-Bloch winding number:

$$W = \frac{i}{2\pi} \int_{C_\beta} q^{-1} dq \tag{46}$$



Note

Crucially, it is defined on the "generalized Brillouin zone" C_β. The image of high-dimensional GBZ is not very clear.

Outline

Non-Hermitian Chern insulator[4]

$$H(\mathbf{k}) = (v_x \sin(k_x) + i\gamma_x)\sigma_x + (v_y \sin(k_y) + i\gamma_y)\sigma_y + (m - t_x \cos(k_x) - t_y \cos(k_y) + i\gamma_z)\sigma_z \quad (47)$$

$$E_\pm(\mathbf{k}) = \pm\sqrt{\sum_{j=x,y,z} (h_j^2 - \gamma_j^2 + 2i\gamma_j h_j)} \quad (48)$$

where $(h_x, h_y, h_z) = (v_x \sin(k_x), v_y \sin(k_y), v_z \sin(k_z))$.
The Bloch bands are gapped if

$$E_\pm(\mathbf{k}) \neq 0 \Rightarrow m > m_+ \quad \text{and} \quad m < m_-$$

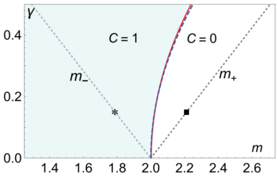
For $\gamma_z = 0$

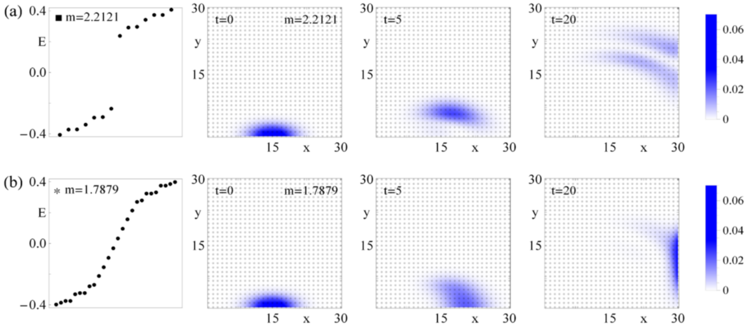
$$m_\pm = t_x + t_y \pm \sqrt{\gamma_x^2 + \gamma_y^2} \quad (49)$$

$H(\mathbf{k})$-based Chern number is nondefinable in the gapless region $m \in [m_-, m_+]$

[4] S.Yao,F.Song,Z.Wang,PRL,121,136802

Topological phase diagram





Non-Bloch Chern number

To describe open-boundary eigenstates

$$\mathbf{k} \to \tilde{\mathbf{k}} + i\tilde{\mathbf{k}}'$$

Define a "non-Bloch Hamiltonian" as follows:

$$\tilde{H}(\tilde{\mathbf{k}}) \equiv H(\mathbf{k} \to \tilde{\mathbf{k}} + i\tilde{\mathbf{k}}')$$

$\tilde{H}(\tilde{\mathbf{k}})$ is generally non-Hermitian, we define the standard right or left eigenvector by

$$\tilde{H}(\tilde{\mathbf{k}})|u_{R\alpha}\rangle = E_\alpha|u_{R\alpha}\rangle \quad \tilde{H}(\tilde{\mathbf{k}})^\dagger|u_{L\alpha}\rangle = E_\alpha^*|u_{L\alpha}\rangle \tag{50}$$

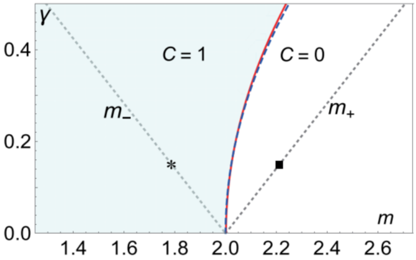
The normalization $\langle u_{L\alpha}|u_{R\alpha}\rangle = 1$ is required in defining Chern numbers.

Non-Bloch Chern number

$$C_\alpha = \frac{1}{2\pi i}\int_{\tilde{T}^2} d^2\tilde{\mathbf{k}}\epsilon^{ij}\langle\partial_i u_{L\alpha}(\tilde{\mathbf{k}})|\partial_j u_{R\alpha}(\tilde{\mathbf{k}})\rangle \tag{51}$$

Focus on the Chern number of "valence band"Re$(E_\alpha < 0)$, compute the Chern number from $\tilde{H}(\tilde{\mathbf{k}})$, when $t_{x,y} = v_{x,y} = 1$, $\gamma_{x,y} = \gamma$, the phase boundary is

$$m = 2 + \gamma^2 \tag{52}$$



Outline

Non-Hermitian Research

- Higher-order non-Hermitian skin effect (PRB,102,205118)
- Non-Hermitian nodal-line semimetal (PRB,99,075130)
- DMFT Reveals the Non-Hermitian Topology and Fermi Arcs in Heavy-Fermion Systems (PRL,125,227204)
- Parity-time-symmetric topological superconductor (PRB,98,085116)
- Defective Majorana zero modes in non-Hermitian Kitaev chain (Arxiv,2010,11451)
- ...